ISRAEL JOURNAL OF MATHEMATICS 114 (1999), 149–156

LINEARIZED OSCILLATION THEOREMS FOR NEUTRAL DIFFERENCE EQUATIONS*

BY

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ABSTRACT

Consider the nonlinear neutral difference equation

$$\Delta[x_n - p_n g(x_{n-k})] + q_n h(x_{n-l}) = 0, \quad n \in N.$$

We establish a linearized oscillation theorem which is a discrete result of the open problem by Gyori and Ladas.

1. Introduction

The linearized oscillation theory for the first order nonlinear neutral differential equation

$$\frac{d}{dt}[x(t) - p(t)g(x(t-\tau))] + q(t)h(x(t-\sigma)) = 0 \tag{1}$$

has been established by Ladas and others [4, 7]. As we have seen in [4, 7], it seems that the following assumption,

$$\limsup_{t\to\infty} p(t) = p_0 \in (0,1), \quad \liminf_{t\to\infty} p(t) = p \in (0,1),$$

is always assumed to hold. Therefore, Gyori and Ladas proposed the following question in [4, Problem 6.12.7]: Obtain linearized oscillation results of Eq. (1)

* This work is partially supported by NNSF (NO: 19671027) of China.
Received February 1, 1998

when the coefficient $p(t) < 0$ for $t \geq t_0$ or $p(t) \geq 1$ for $t \geq t_0$. The linearized oscillation results of Eq. (1) have been obtained by Chen and Yu in [2] for the case where $\limsup_{t\to\infty} p(t) = p_0 \in (1,\infty)$, and by Yu and Wang in [8] for the case where $\lim_{t\to\infty} p(t) = p_0 \in (-\infty,-1)$. But there are no results for the case where $\lim_{t\to\infty} p(t) = 1$, therefore we establish a discrete result for this case.

Let Z denote the set of all integers. For given $a, b \in Z$ with $a \leq b$, define $N(a) = \{a, a+1, \ldots\}, N = N(0)$ and $N(a,b) = \{a, a+1, \ldots, b\}$.

Consider the difference equation

$$\triangle[x_n - p_n g(x_{n-k})] + q_n h(x_{n-l}) = 0, \quad n \in N, \tag{2}$$

where $\triangle$ denotes the forward difference operator defined by $\triangle x_n = x_{n+1} - x_n$, $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers,

$$k, l \in N, \quad k \geq 1, \quad g, h \in C[R, R]. \tag{3}$$

Moreover, we assume that

$$\lim_{n\to\infty} p_n = 1, \quad \lim_{n\to\infty} q_n = q \in (0,\infty), \tag{4}$$

$$ug(u) > 0, \quad \text{for } u \neq 0, \quad \lim_{u\to 0} \frac{g(u)}{u} = 1, \quad \lim_{|u|\to\infty} \frac{g(u)}{u} = 1, \tag{5}$$

$$uh(u) > 0, \quad \text{for } u \neq 0, \quad \lim_{u\to 0} \frac{h(u)}{u} = 1, \quad \lim_{|u|\to\infty} \frac{h(u)}{u} = 1. \tag{6}$$

We obtain the following theorem which is a discrete result of Eq. (1) for the case where $\lim_{t\to\infty} p(t) = 1$.

THEOREM 1: *If (3), (4), (5) and (6) hold, then every solution of Eq. (2) oscillates.*

As an application of Theorem 1, we consider the following difference equation:

$$\triangle[x_n - p_n g(x_{n-1})] + q_n h(x_{n-2}) = 0, \tag{7}$$

where

$$p_n = \frac{n^2+1}{(n^2+1) + (-1)^n n}, \quad q_n = \frac{2(2n^2+4n+1)(n-1)}{n(n+1)(n+2)},$$

$$g(x) = \frac{x(1+x+x^2)}{1+x^2}, \quad h(x) = x.$$

It is easy to see that all the assumptions of Theorem 1 are satisfied with $q = 4$. Therefore, every solution of Eq. (7) oscillates. In fact, $x_n = (-1)^{n+1}/(n+1)$ is such a solution.

For the linearized oscillations of difference equations, we refer to [3, 6, 9].

Set $m = \max\{k, l\}$. By a solution of Eq. (2), we mean a sequence $\{x_n\}$ of real numbers which is defined for all $n \in N(n_0 - m)$ and satisfies Eq. (2) for $n \in N(n_0)$, for some $n_0 \in N$. A solution $\{x_n\}$ is said to be **oscillatory** if the terms x_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called **nonoscillatory**. For the general background on difference equations, one can refer to [1, 5].

2. Proof of Theorem 1

Before proving the theorem, we establish two lemmas which are useful in the proof of Theorem 1.

Consider the linear difference equation

$$\triangle(x_n - px_{n-k}) + qx_{n-l} = 0, \quad n \in N, \tag{8}$$

where

$$k, l \in N, \quad k \geq 1, \quad p, q \in (0, \infty). \tag{9}$$

LEMMA 1: *Assume (9) holds and either*

$$0 < p \leq 1, \quad q > (1 - p^{1/k})(1 - p), \tag{10}$$

or

$$p > 1, \quad q > (p^{1/k} - 1)(p - 1)p^{l/k}. \tag{11}$$

Then every solution of Eq. (8) oscillates.

Proof: It is well known [4] that every solution of Eq. (8) oscillates if and only if the characteristic equation of Eq. (8)

$$(\lambda - 1)(1 - p\lambda^{-k}) + q\lambda^{-l} = 0 \tag{12}$$

has no real roots. It is easy to see, by a simple computation, if (10) or (11) holds; then (12) has no real roots. This completes the proof of Lemma 1. ∎

The following lemma can be found in [9, Lemma 2.1].

LEMMA 2: *Assume that $k, l \in N$, $k \geq 1$, $\{p_n\}, \{q_n\}$ are nonnegative, and there is a positive integer k^* such that*

$$p_{k^*+ik} \leq 1, \quad i \in N.$$

If the difference inequality

$$\triangle(y_n - p_n y_{n-k}) + q_n y_{n-l} \leq 0, \quad n \in N$$

has an eventually positive solution, then the corresponding difference equation

$$\triangle(x_n - p_n x_{n-k}) + q_n x_{n-l} = 0, \quad n \in N$$

has an eventually positive solution.

Proof of Theorem 1: Assume, for the sake of contradiction, that Eq. (2) has a nonoscillatory solution $\{x_n\}$. We only consider the case where $\{x_n\}$ is eventually positive. The case where $\{x_n\}$ is eventually negative is similar and the proof will be omitted. Assume $x_n > 0$ for $n \in N(n_0 - m)$; let

$$z_n = x_n - p_n g(x_{n-k}). \tag{13}$$

Then

$$\triangle z_n = -q_n h(x_{n-l}) \leq 0, \quad n \in N(n_0), \tag{14}$$

which implies that $\{z_n\}_{n \in N(n_0)}$ is nonincreasing. There are two possible cases to consider.

CASE 1: $\lim_{n\to\infty} z_n = z^* \in R$.

In this case, we know from (14) that

$$\lim_{n\to\infty} q_n h(x_{n-l}) = \lim_{n\to\infty} (- \triangle z_n) = 0.$$

Since $\lim_{n\to\infty} q_n = q > 0$, we see that $\lim_{n\to\infty} h(x_{n-l}) = 0$. In view of (6), we are led to

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} z_n = 0. \tag{15}$$

Define

$$P_n = \frac{p_n g(x_{n-k})}{x_{n-k}}, \quad Q_n = \frac{q_n h(x_{n-l})}{x_{n-l}};$$

then

$$(16)\quad \lim_{n\to\infty} P_n = \lim_{n\to\infty} p_n \cdot \lim_{u\to 0} \frac{g(u)}{u} = 1, \ \lim_{n\to\infty} Q_n = \lim_{n\to\infty} q_n \cdot \lim_{u\to 0} \frac{h(u)}{u} = q > 0,$$

and Eq. (2) reduces to

$$(17)\qquad \triangle(x_n - P_n x_{n-k}) + Q_n x_{n-l} = 0.$$

By (13), (17) can be rewritten as

$$(18)\qquad \triangle z_n - P_{n-l}\frac{Q_n}{Q_{n-k}} \triangle z_{n-k} + Q_n z_{n-l} = 0.$$

Choosing $p \in (0,1), q_0 = q/2$ such that

$$(19)\qquad (1-p)(1-p^{1/k}) < q_0,$$

then there exists $n^* \in N(n_0)$ such that

$$(20)\qquad P_{n-l}\frac{Q_n}{Q_{n-k}} \geq p, \quad Q_n \geq q_0 \quad \text{for } n \in N(n^* - m).$$

Since $z_n > 0$, $\triangle z_n \leq 0$ $(n \in N(n_0))$, (18) yields

$$(21)\qquad \triangle z_n - p \triangle z_{n-k} + q_0 z_{n-l} \leq 0, \quad n \in N(n^*).$$

In view of Lemma 2, the corresponding equation

$$(22)\qquad \triangle z_n - p \triangle z_{n-k} + q_0 z_{n-l} = 0$$

has an eventually positive solution, while from (19) and Lemma 1 we see that every solution of Eq. (22) oscillates. This is a contradiction.

CASE 2: $\lim_{n\to\infty} z_n = -\infty$.
In this case, we have

$$(23)\qquad \lim_{n\to\infty} x_n = \infty.$$

In fact, if (23) does not hold, then there is a sequence of integers $\{n_i\}$ such that

$$\lim_{i\to\infty} n_i = \infty, \quad \lim_{i\to\infty} x_{n_i} = x^* \in R.$$

Thus

$$z_{n_i+k} = x_{n_i+k} - p_{n_i+k} g(x_{n_i}), \quad \liminf_{i\to\infty} z_{n_i+k} \geq -g(x^*);$$

this is impossible. So (23) holds.

Let

$$P_n^* = \frac{p_n g(x_{n-k})}{x_{n-k}}, \quad Q_n^* = \frac{q_n h(x_{n-l})}{x_{n-l}};$$

then

$$\lim_{n\to\infty} P_n^* = \lim_{n\to\infty} p_n \cdot \lim_{u\to\infty} \frac{g(u)}{u} = 1, \ \lim_{n\to\infty} Q_n^* = \lim_{n\to\infty} q_n \cdot \lim_{u\to\infty} \frac{h(u)}{u} = q. \tag{24}$$

Choose $p_0 \in (1, \infty)$, $q_0 = q/2$ such that

$$(p_0^{1/k} - 1)(p_0 - 1)p_0^{l/k} < q_0. \tag{25}$$

In view of (24), there is $m^* \in N(n_0)$ such that

$$z_{m^*} < 0, \quad P_n^* < p_0 \quad \text{and} \quad Q_n^* > q_0 \quad \text{for } n \in N(m^*). \tag{26}$$

By the definition of P_n^*, Q_n^*, we see that

$$\triangle(x_n - P_n^* x_{n-k}) + Q_n^* x_{n-l} = 0.$$

Summing both sides of this equation from m^* to $n-1$, we get

$$x_n - P_n^* x_{n-k} - z_{m^*} + \sum_{i=m^*}^{n-1} Q_i^* x_{i-l} = 0, \quad n \in N(m^*+1). \tag{27}$$

From (26), this implies

$$x_n - p_0 x_{n-k} - z_{m^*} + q_0 \sum_{i=m^*}^{n-1} x_{i-l} \le 0, \quad n \in N(m^*+1). \tag{28}$$

So

$$x_n \ge (x_{n+k} - z_{m^*} + q_0 \sum_{i=m^*}^{n+k-1} x_{i-l})/p_0, \quad n \in N(m^*-k+1). \tag{29}$$

Define the set of positive sequences

$$S = \{a = \{a_n\}\colon -z_{m^*}/p_0 \le a_n \le x_n, \quad n \in N(m^*-k+1)\}, \tag{30}$$

and the mapping T on S as follows: For every $a = \{a_n\} \in S$, $b = Ta = \{b_n\}$ where

$$b_n = \begin{cases} a_n, & n \in N(m^*-k+1, m^*-k+l), \\ (a_{n+k} - z_{m^*} + q_0 \sum_{i=m^*}^{n+k-1} a_{i-l})/p_0, & n \in N(m^*-k+l+1). \end{cases} \tag{31}$$

It is clear that T is monotone in the sense that if $a, b \in S$, $a \leq b$ (that is $a_n \leq b_n$ for $n \in N(m^* - k + 1)$), then $Ta \leq Tb$. Let $x = \{x_n\}_{n \in N(m^*-k+1)}$; (29) implies $Tx \leq x$, so $a \in S$ implies $Ta \leq Tx \leq x$. Obviously, $(Ta)_n \geq -z_{m^*}/p_0$. Therefore $T\colon S \to S$.

We now define the following sequence on S:

$$y^{(0)} = x \quad \text{and} \quad y^{(s)} = Ty^{(s-1)} \quad \text{for } s = 1, 2, \ldots.$$

By induction, we know that the sequence $\{y^{(s)}\}$ of elements of S satisfies

$$-z_{m^*}/p_0 \leq y_n^{(s+1)} \leq y_n^{(s)} \leq x_n \quad \text{for } n \in N(m^* - k + 1).$$

Hence

$$y_n = \lim_{s \to \infty} y_n^{(s)}, \quad n \in N(m^* - k + 1)$$

exists and $y = \{y_n\}_{n \in N(m^*-k+1)} \in S$. Also $Ty = y$, that is

$$y_n = (y_{n+k} - z_{m^*} + q_0 \sum_{i=m^*}^{n+k-1} y_{i-l})/p_0, \quad n \in N(m^* - k + 1),$$

which is to say that $\{y_n\}$ is a positive solution of the difference equation

$$\triangle(x_n - p_0 x_{n-k}) + q_0 x_{n-l} = 0, \quad n \in N(m^* + 1). \tag{32}$$

However, from (25) and Lemma 1, we see that every solution of Eq. (32) oscillates; this is a contradiction.

The proof of Theorem 1 is complete by combining Case 1 and Case 2. ∎

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